Communication for maths

On the formal presentation of differentiation – part 1

Introduction

- These slides illustrate certain aspects relating to the presentation of maths.
- Some are general, and relate to the presentation of all maths. Some are specific to the presentation of differentiation.
- Not all aspects that we have studied so far will be presented here.
- Revise the slides on the previous topics where appropriate.

Terminology

- Functions and limits are the most fundamental things in differentiation (and integration).

- It is important to be able to
  - use the correct terminology when speaking about these,

  and

  - correctly represent the symbols for these.

Terminology

1) An **equation** is a statement that two things are equal.

a) $\frac{df}{dx} = 3x^2 - 1$ This is an equation

b) $3x^2 - 1$ This is not an equation

Terminology

2) An **expression** is a valid string of mathematical symbols.

a) $$\frac{f(x+\delta x)-f(x)}{\delta x}$$ This is an expression

b) $$\frac{(\quad)f(-fx)}{x\delta}$$ This is not an expression

Terminology

3) A **function** is a process which transforms an input value into an output value according to a certain rule.

4) Other terminology for “function” are “mapping” or “transformation”

Terminology

5) Symbolism:

- The symbol f denotes a function. The symbol $f(3)$ does not.
- The symbol $f(3)$ denotes a value: the evaluation of the function (say) $x = 3$.
- Therefore $f(x)$ does not represent a function, but the evaluation of the function at an arbitrary point x.

Terminology

6) Contradiction: However, it has become common for people to refer to $f(x)$ as a function.

7) A function is a process not an equation. So

$$f(x) = 3x^2 + 1$$

is not a function. It is an equation which defines a rule, or formula, based on a variable x, for the function.

Terminology

8) The function f, defined by the equation $f(x) = 3x^2 + 1$, is actually the process of transforming x by squaring it, then multiplying by 3 then adding 1.

Terminology

- **Example 1:** What is wrong with the following?

Let $f(x) = x^2$. In

$$h(x) = f(2x - 1) + f(x + 3)$$

we have substituted into the first function $2x - 1$ in the equation of f, and substituted $x + 3$ into the second function in the equation of f.

Answer: See lesson

Terminology

- **Example 2:** What is wrong with the following?

Given the function $x^2 + 1$ evaluate for the function $f(4)$.

Answer: See lesson

Terminology

- **Example 3:** What is wrong with the following?

Given the function $x^2 + 1$ solve for $f(4)$.

Answer: See lesson

Quote

- The following quote from H. P. Thielman illustrates that maths is not just about symbols. There can be a lot of writing in maths.

- It also illustrates the importance in mathematics of defining terms, so that everybody knows what any given term means. This then allows us to use the term correctly and without confusion.

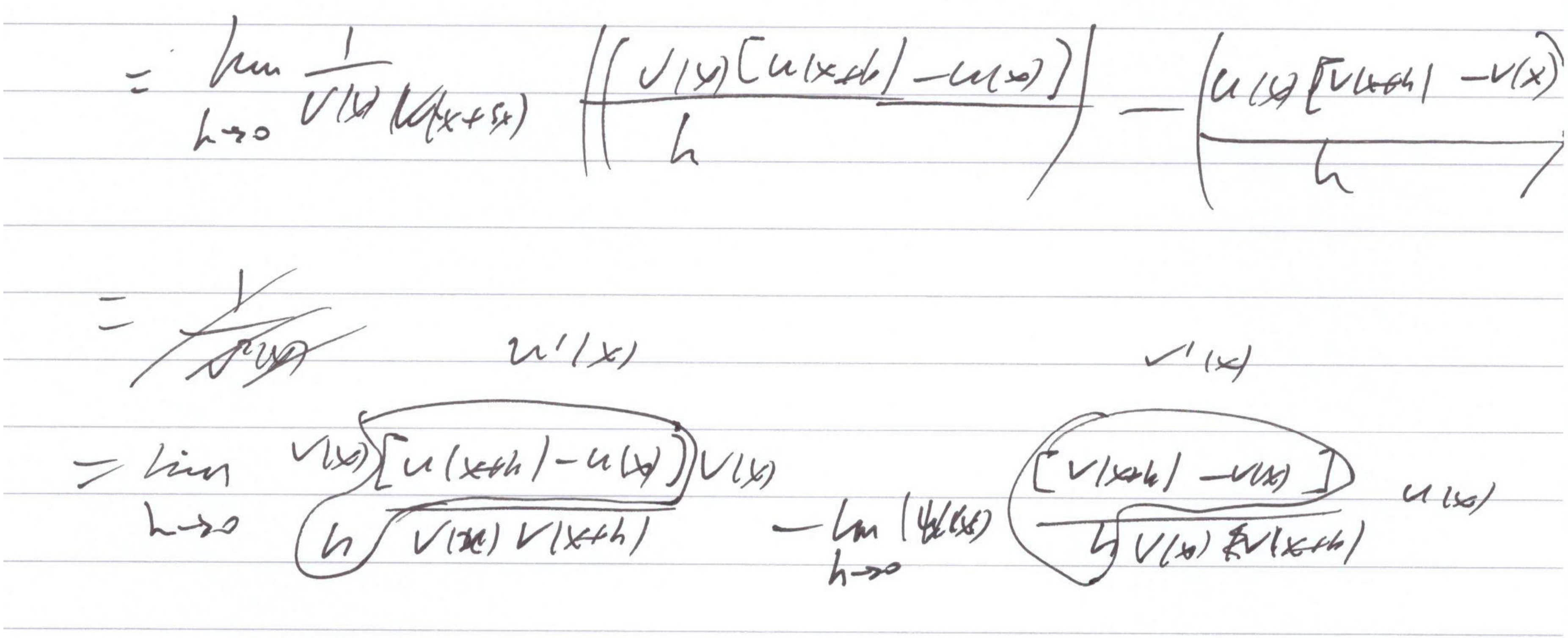
Quote

Let f be a given function with domain of definition X, and with range Y. If x stands for an unspecified element of X, x is called the *independent variable* of the given function. If y stands for an unspecified element of the range Y, then y is called the *dependent variable* of the given function. For a given ordered pair (x, y) of f, y is called the *image* of x under f, while x is called the *counter image* or *source* of y under f. The image of x under f is also called the *value of the function f at x*, and is denoted by $f(x)$. A function f whose domain of definition is X, and whose range is Y is frequently denoted by $f: X \rightarrow Y$, and is referred to as a function *on* X *onto* Y.

"On the Definition of Functions", H. P. Thielman, *The American Mathematical Monthly*, Vol. 60, No. 4 (Apr., 1953), pp. 259-262

Clean and readable presentation

Example



This is not readable

Implication symbol

- How to use the symbol "⇒"

given $y = e^x \cdot \sin x$

Then $y' \Rightarrow e^x \cdot \sin x + e^x \cdot \cos x$ **No**

Implication symbol

- **How to use the symbol "⇒"**

given $y = e^x \cdot \sin x$

Then $y' = e^x \cdot \sin x + e^x \cdot \cos x$

$\Rightarrow \quad y'' = e^x \cdot \sin x + e^x \cdot \cos x + e^x \cdot \cos x - e^x \cdot \sin x$ **Yes**

$y'' = 2e^x \cdot \cos x$

$\Rightarrow \quad y'' - 2y' + 2y = 0$ **Yes**

Write in complete sentences and start each sentence with a word.

- Start each sentence with a word not a symbol.

- *Example*
  - "f is a differentiable function". **No**
  - "Function f is differentiable". **Yes**

or better still

  - "Let f be a differentiable function". **Yes**

Write in complete sentences and start each sentence with a word.

- Start each sentence with a word not a symbol.

- <u>*Example*</u>
  - "df/dx is the derivative of $y = f(x)$". **No**
  - "Function y has derivative dy/dx". **Yes**

or better still

  - "The derivative of $y = f(x)$ is df/dx". **Yes**

Correct use of symbols

The definition of dy/dx

The following illustrate the correct presentation of the definition of the first derivative:

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$\frac{df}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Correct use of symbols

What is wrong with the following?

$$\frac{dy}{dx} = \frac{f(x+h) - f(x)}{\delta x}$$

Correct use of symbols

What is wrong with the following?

$$\frac{dy}{dx} = \text{lim} = \frac{f(x+h) - f(x)}{\delta x}$$

Correct use of symbols

What is wrong with the following?

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{\delta x}$$

Correct use of symbols

What is wrong with the following?

$$\frac{dy}{dx} = L \, \frac{1}{h} \left(f(x+h) - f(x) \right)$$

Correct use of symbols

What is wrong with the following?

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{1}{h} \cdot \left(f(x+h) - f(x) \right)$$

Correct use of symbols

What is wrong with the following?

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{\left(\frac{1}{(x+h)^2} - \frac{1}{x^2} \right)}{h}$$

Correct use of symbols

What is wrong with the following?

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{1}{h} \cdot \frac{1}{(x+h)^2} - \frac{1}{x^2}$$

Correct use of symbols

What is wrong with the following?

$$y' = \sqrt{\lim_{h \to 0}}\ \frac{1}{h}\left(\frac{1}{(x+h)^2} - \frac{1}{x^2}\right)$$

Correct use of symbols

Writing

$$\frac{dy}{dx} = \lim_{h \to 0} \left(\frac{\frac{1}{(x+h)^2} - \frac{1}{x^2}}{h} \right)$$

in this way according to this format should be so automatic that you should not have to think about having to write it this way.

Practice until it becomes ingrained.

Correct use of symbols

Examples: The following illustrate the correct use and presentation of symbols:

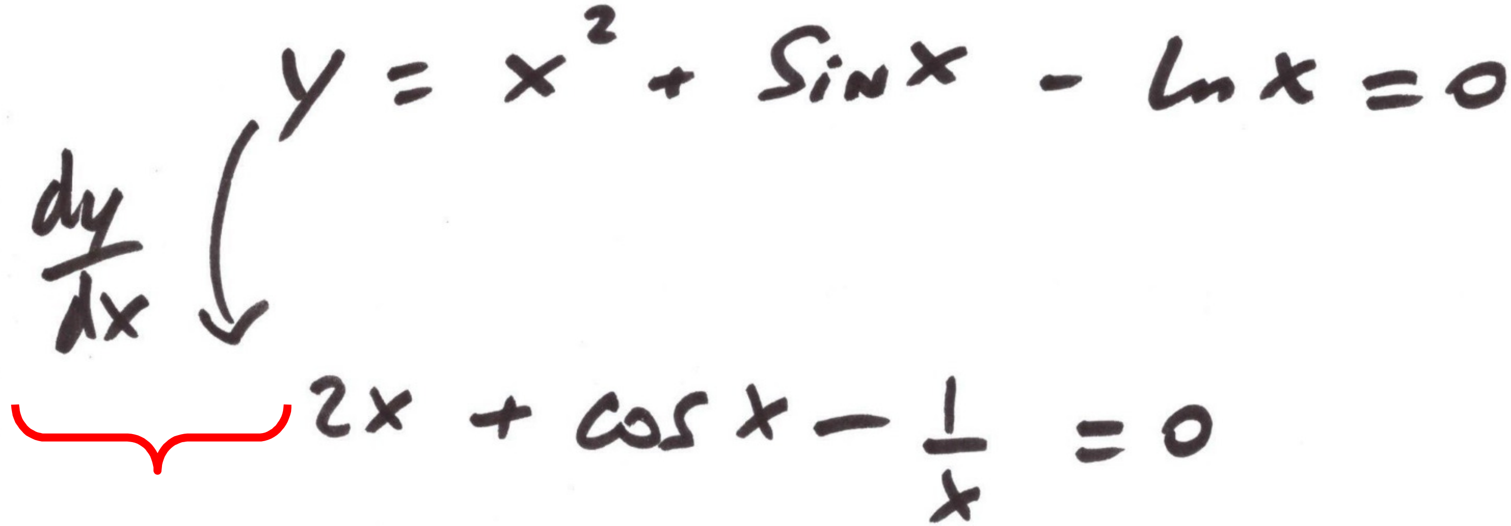
1) $$f'(x) = y' = \frac{df}{dx} = \frac{dy}{dx} = \frac{d}{dx}(f(x)) = \frac{d}{dx}(y)$$

2) $$f'(a) = y'\big|_{x=a} = \left.\frac{df}{dx}\right|_{x=a} = \left.\frac{dy}{dx}\right|_{x=a}$$

3) $$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

Correct use of symbols

No arrows (unless it is part of the mathematics)



No

Correct use of symbols

No arrows (unless it is part of the mathematics)

$$y = x^2 + \sin x - \ln x = 0$$

$$\therefore$$

$$\frac{dy}{dx} = 2x + \cos x - \frac{1}{x} = 0$$

Yes

Justify steps where relevant

Example:

The example handed out show the proof of

$$\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1.$$

What is wrong with the presentation of the proof?

Justify steps where relevant

Exercise: What justifying step(s) do you think you would need to include in your solutions to questions on the following?

- Finding the first derivative of a function;
- Using the quotient rule for differentiation;
- Finding the tangent and normal to a function at a given point;

Example and exercise

Example 1: Consider the example handed out on how to prove $(u.v)' = u'.v + u.v'$, where u and v are functions of x. What errors of mathematical communication are missing from this example?

Exercise 1: What errors of mathematical communication are missing from the question handed out on differentiation from first principles?

===============

Appendix

===============

Correct use of symbols

Example: *(*used as exercise 1 above*)*

Using first principles show that the first derivative

$f(x) = x^2$ is

$$\frac{df}{dx} = 2x$$

Correct use of symbols

Example

What is wrong with this solution?

$$\lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} = \lim \frac{x^2 + 2hx + h^2 - x^2}{h}$$

$$\lim_{h \to 0} \frac{2hx + h^2}{h} \Longrightarrow \lim 2x + h = 2x$$

Justify steps where relevant

Example

Using first principles show that the first derivative of $f(x) = x^2$ is

$$\frac{df}{dx} = 2x$$

Justify steps where relevant

Example: The solution below does not show enough steps:

Solution:

$$\frac{df}{dx} = \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h}$$

$$= \lim_{h \to 0} \frac{2hx + h}{h} = 2x$$

Justify steps where relevant

Example:

The solution here shows enough steps.

Also, notice the alignment of the "=" symbol

$$\frac{df}{dx} = \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h}$$

$$= \lim_{h \to 0} \frac{x^2 + 2hx + h^2 - x^2}{h}$$

$$= \lim_{h \to 0} \frac{2hx + h^2}{h}$$

$$= \lim_{h \to 0} (2x + h) \quad = 2x$$

Justify steps where relevant

Example: For a problem about finding the equations of tangents and normals to a given function what justifying step(s) do you think you would need to include?

Indicative answer:

1. The answer to the gradient of the tangent

$$\frac{dy}{dx} = \cdots$$

Justify steps where relevant

Indicative answer:

2. A statement about the gradient of the normal compared to that of the tangent (what is this statement?);

3. The final answer written as “The equation of the … is …”.

   – In other words, answer the question.